KEY

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MCS 231
Linear Algebra
2^{nd} Midterm
December 17, 2009
08:40-10:20

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 5 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

1. The coefficient matrix of a certain 4×5 homogeneous linear system is

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

a) Find a basis β for its solution space.
b) What is the nullity and rank of A?
c) Find the vector α, where $[\alpha]_\beta = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.
d) Determine whether or not $\gamma = (-4, 5, -2, -2, 0)$ is a solution of the system.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow[-R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{\frac{1}{2}R_4 \to R_4} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$x_2 = r, \quad x_4 = t$

$x_1 = -2r - 3t$
$x_2 = r$
$x_3 = t$
$x_4 = t$
$x_5 = 0$

$$\begin{bmatrix} -2r-3t \\ r \\ t \\ t \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{V_1} r + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{V_2} t, \quad r, t \in \mathbb{R}$$

4 $\quad \beta = \{V_1, V_2\}$

basis for the solution space (Nullspace)

b) Nullity(A) = dimension of the nullspace = 2 2
Rank(A) = $n -$ nullity(A) $= 5 - 2 = 3$ 2

c) $[\alpha]_\beta = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \Rightarrow \alpha = 5 \cdot V_1 - 2V_2$ 2

$\alpha = 5(-2,1,0,0,0) - 2(-3,0,1,1,0)$

$\alpha = (-4,5,-2,-2,0)$ 2

d) Since $\gamma = (-4,5,-2,-2,0) = 5V_1 - 2V_2 \Rightarrow$ it is a solution of the system. 4

(it can be written linear combination of the basis vectors of the solu. space)

2. Consider the sest $S=\{\overset{V_1}{2x^2+x+1}, \overset{V_2}{x^2+3x+2}, \overset{V_3}{-x^2+2x+1}, \overset{V_4}{4x^2-3x-1}\}$.

a) Find a basis β for Span (S) consisting of vectors of S.

b) Express each vector not in the basis as a linear combination of basis vectors.

c) Find a basis for P_2 containing β.

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a)

$$\begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 3 & 2 & -3 \\ 2 & 1 & -1 & 4 \end{bmatrix} \text{ (2) } \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -3 & -3 & 6 \end{bmatrix} \text{ (2) } \xrightarrow{3R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (1) }$$

$$\xrightarrow{-2R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (1) }$$

$\beta=\{V_1, V_2\}=\{2x^2+x+1, 2+3x+x^2\}$ (2)

basis for span(S).

b) $V_3=-x^2+2x+1$ $\qquad V_3=-V_1+V_2$ (2)

$V_4=4x^2-3x-1$ $\qquad V_4=3V_1-2V_2$ (2)

c) $\text{Span}(e_1, e_2, e_3)=P_2$ (2)

$$\begin{array}{ccccc} V_1 & V_2 & e_1 & e_2 & e_3 \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix} \text{ (2) } \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -3 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{3R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -5 & 3 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_3\to R_3} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & -1/5 \end{bmatrix} \text{ (2) }$$

So, V_1, V_2 and e_1 form a basis for P_2.

$\{2x^2+x+1, x^2+3x+2, 1\}$ (2)

$$\begin{array}{cccccc} & \alpha_1 & \alpha_2 & \alpha_3 & & \alpha_1' \quad \alpha_2' \quad \alpha_3' \end{array}$$

3. Consider the bases $\beta = \{(1,1,1),(1,2,2),(2,3,4)\}$ and $\beta' = \{(4,6,7),(0,1,1),(0,1,2)\}$

a) Show that β and β' are bases for $\mathbb{R}^3$.

b) Find the vector α, where $[\alpha]_{\beta'} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

c) Find the transition matrix P from β' to β.

d) Find $[\alpha]_\beta$ using P.

e) Find $[\alpha]_\beta$ directly.

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a) 4

$$\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 1.(8-6)-1(4-3)+2(0) = 1 \neq 0$$

$$\begin{vmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{vmatrix} = 4 \neq 0 \Rightarrow \beta \text{ and } \beta' \text{ are bases for } \mathbb{R}^3$$

$$\begin{array}{ccc} \alpha_1' & \alpha_2' & \alpha_3' \end{array}$$

(they are lin. indep and span $\mathbb{R}^3$)

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b) $[\alpha]_{\beta'} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \Rightarrow \alpha = 2.\alpha_1' + 3\alpha_2' - 4\alpha_3'$

$\alpha = 2.(4,6,7) + 3(0,1,1) - 4(0,1,2)$

$\alpha = (8,11,9)$

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c) $P = \left[[\alpha_1']_\beta \,\vdots\, [\alpha_2']_\beta \,\vdots\, [\alpha_3']_\beta \right]$

$$\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1' & \alpha_2' & \alpha_3' \end{array}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 4 & 0 & 0 \\ 1 & 2 & 3 & 6 & 1 & 1 \\ 1 & 2 & 4 & 7 & 1 & 2 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 & 2 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[-R_3+R_2 \to R_2]{-2R_3+R_1 \to R_1} \left[\begin{array}{ccc|c|c|c} 1 & 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

P

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d) $[\alpha]_\beta = P.[\alpha]_{\beta'} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$

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e)

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 1 & 2 & 3 & 11 \\ 1 & 2 & 4 & 9 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[-R_3+R_2 \to R_2]{-2R_3+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$[\alpha]_\beta = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$$

4. Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 3a+b-c \\ a+b+7c \end{bmatrix}.$$

and $S = \left\{ \overset{V_1}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}, \overset{V_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}, \overset{V_3}{\begin{bmatrix} 1 \\ -3 \\ -7 \end{bmatrix}} \right\}$ is a basis for $\mathbb{R}^3$ and $T = \left\{ \overset{W_1}{\begin{bmatrix} 2 \\ 7 \end{bmatrix}}, \overset{W_2}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} \right\}$ is a basis for $\mathbb{R}^2$.

a) Show that L is a linear transformation.

b) Find the representation matrix ~~of~~ A with respect to S and T.

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a) Let $\alpha = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\beta = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \in \mathbb{R}^3$

$$L(\alpha+\beta) = L\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix} = \begin{bmatrix} 3(a_1+a_2)+(b_1+b_2)-(c_1+c_2) \\ (a_1+a_2)+(b_1+b_2)+7(c_1+c_2) \end{bmatrix}$$

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$$= \begin{bmatrix} 3a_1+b_1-c_1 \\ a_1+b_1+7c_1 \end{bmatrix} + \begin{bmatrix} 3a_2+b_2-c_2 \\ a_2+b_2+7c_2 \end{bmatrix} = L(\alpha)+L(\beta)$$

Let $c \in \mathbb{R}$, $L(c\alpha) = \begin{bmatrix} ca_1 \\ cb_1 \\ cc_1 \end{bmatrix} = \begin{bmatrix} 3(ca_1)+(cb_1)-(cc_1) \\ (ca_1)+(cb_1)+7(cc_1) \end{bmatrix} = c\begin{bmatrix} 3a_1+b_1-c_1 \\ a_1+b_1+7c_1 \end{bmatrix}$

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$$= cL(\alpha)$$

$\Longrightarrow$ L is a linear transformation.

b) $A = \left[[L(V_1)]_T \;\vdots\; [L(V_2)]_T \;\vdots\; [L(V_3)]_T \right]$ 2

$$L(V_1) = L\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3.2+1+1 \\ 2+1-7 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \quad 1$$

$$L(V_2) = L\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3.0+1-2 \\ 0+1+7.2 \end{bmatrix} = \begin{bmatrix} -1 \\ 15 \end{bmatrix} \quad 1$$

$$L(V_3) = L\begin{bmatrix} 1 \\ -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 3-3+7 \\ 1-3-49 \end{bmatrix} = \begin{bmatrix} 7 \\ -51 \end{bmatrix} \quad 1$$

$$\begin{array}{c} \begin{array}{ccccc} W_1 & W_2 & L(V_1) & L(V_2) & L(V_3) \end{array} \\ \left[\begin{array}{cc|c|c|c} 2 & 1 & 8 & -1 & 7 \\ 7 & 4 & -4 & 15 & -51 \end{array}\right] \end{array} \xrightarrow[1]{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c|c|c} 1 & 1/2 & 4 & -1/2 & 7/2 \\ 7 & 4 & -4 & 15 & -51 \end{array}\right] \xrightarrow{-7R_1+R_2 \to R_2}$$

$$\left[\begin{array}{cc|c|c|c} 1 & 1/2 & 4 & -1/2 & 7/2 \\ 0 & 1/2 & -32 & 37/2 & -15/2 \end{array}\right] \xrightarrow[2R_2 \to R_2]{-R_2+R_1 \to R_1} \left[\begin{array}{cc|c|c|c} 1 & 0 & 36 & -19 & 79 \\ 0 & 1 & -64 & 37 & -151 \end{array}\right]$$

So, $A = \begin{bmatrix} 36 & -19 & 79 \\ -64 & 37 & -151 \end{bmatrix}$ 4

5. Let $L: P_2 \longrightarrow P_2$ be a linear operator defined by

$$L\left(a+bx+cx^2\right)=(a+c)+(a+b+2c)x+(2a+b+3c)x^2$$

a) Find a basis for $\ker(L)$ and the nullity of L.
b) Find a basis for Range(L) and the rank of L.
c) Is L invertible? Explain.

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a) $\alpha = a+bx+cx^2 \in Ker(L) \Rightarrow L(\alpha)=0_{P_2}$ (1)

$L(a+bx+cx^2) = (a+c)+(a+b+2c)x+(2a+b+3c)x^2 = 0$

$$\left.\begin{array}{l} a+c=0 \\ a+b+2c=0 \\ 2a+b+3c=0 \end{array}\right\} \Rightarrow \begin{array}{ccc} a & b & c \end{array}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3\to R_3} \begin{array}{ccc} a & b & c \end{array}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ (1) (1) (1)

$c=t$, $a=-t$, $b=-t$ (1)

If $\alpha = a+bx+cx^2 \in Ker(L) \Rightarrow \alpha = -t-tx+tx^2 = t(-1-x+x^2)$, $t\in\mathbb{R}$

(1) basis for $Ker(L)$

OR. $\alpha = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t$, $t\in\mathbb{R}$

$\to$ basis for $Ker(L)$ so nullity$(L)=1$ (2)

b) If $\beta = m+nx+kx^2 \in Range(L)$ then there exist $\alpha = a+bx+cx^2 \in P_2$ such that $L(\alpha)=\beta \Rightarrow L(a+bx+cx^2)=(a+c)+(a+b+2c)x+(2a+b+3c)x^2 = \beta$

$(a+c)+(a+b+2c)x+(2a+b+3c)x^2 = m+nx+kx^2$ (1)

$$\begin{array}{l} a+c=m \\ a+b+2c=n \\ 2a+b+3c=k \end{array} \text{(1)} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & m \\ 1 & 1 & 2 & n \\ 2 & 1 & 3 & k \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & m \\ 0 & 1 & 1 & n-m \\ 0 & 1 & 1 & k-2m \end{array}\right]$$ (1)

$$\xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & m \\ 0 & 1 & 1 & n-m \\ 0 & 0 & 0 & k-m-n \end{array}\right]$$ (1) $\Rightarrow \beta \in Range(L) \Rightarrow k-m-n=0$, $k=m+n$

$\beta = m+nx+kx^2 = m+nx+(m+n)x^2 = \underbrace{(1+x^2)m+(x+x^2)n}_{\text{basis for Range}(L)}$ (2)

OR $\beta = \begin{bmatrix} m \\ n \\ k \end{bmatrix} = \begin{bmatrix} m \\ n \\ m+n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} m + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} n}_{\text{basis for Range}(L)}$, $m,n\in\mathbb{R}$

so, Rank$(L)=2$ (2)

c) No, Since L is not 1-1 (nullity$(L)=1\neq 0$) (4)
L is not onto (rank$(L)=2\neq 3=\dim P_2$)