KEY

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MCS 231
Linear Algebra

1^{st} Midterm

November 10, 2009
17:40-19:20

Surname : ____________________

Name : ____________________

ID # : ____________________

Department : ____________________

Section : ____________________

Instructor : ____________________

Signature : ____________________

- The exam consists of 5 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

1. Solve the following system of linear equations by using Gauss-Jordan Elimination.

$$
\begin{aligned}
x_1 + x_2 + 2x_3 + 3x_4 + x_5 &= 1\\
-3x_1 + x_2 - 18x_3 - 5x_4 + 4x_5 &= 4\\
-x_1 + x_2 - 8x_3 - x_4 &= 0\\
x_1 + 2x_2 - x_3 + 4x_4 - 5x_5 &= -5
\end{aligned}
$$

$$
\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 1 & 1\\ -3 & 1 & -18 & -5 & 4 & 4\\ -1 & 1 & -8 & -1 & 0 & 0\\ 1 & 2 & -1 & 4 & -5 & -5 \end{array}\right]
\xrightarrow[\substack{R_1+R_3\to R_3\\ -R_1+R_4\to R_4}]{3R_1+R_2\to R_2}
\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 1 & 1\\ 0 & 4 & -12 & 4 & 7 & 7\\ 0 & 2 & -6 & 2 & 1 & 1\\ 0 & 1 & -3 & 1 & -6 & -6 \end{array}\right]
$$

$$
\xrightarrow[\substack{-2R_4+R_3\to R_3\\ R_4 \leftrightarrow R_2}]{-4R_4+R_2\to R_2}
\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 1 & 1\\ 0 & 1 & -3 & 1 & -6 & -6\\ 0 & 0 & 0 & 0 & 13 & 13\\ 0 & 0 & 0 & 0 & 31 & 31 \end{array}\right]
\xrightarrow[\substack{\frac{1}{31}R_4\to R_4\\ -R_3+R_4\to R_4}]{\frac{1}{13}R_3\to R_3}
\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 1 & 1\\ 0 & 1 & -3 & 1 & -6 & -6\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\xrightarrow[6R_3+R_2\to R_2]{-R_3+R_1\to R_1}
\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 0 & 0\\ 0 & 1 & -3 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\xrightarrow{-R_2+R_1\to R_1}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\
\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 2 & 0 & 0\\ 0 & 1 & -3 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{array}
$$

(12)

x_1, x_2 and x_5 are basic variables

x_3 and x_4 are parameters,

$x_3 = t$, $x_4 = u$

$$
\begin{aligned}
x_1 + 5x_3 + 2x_4 &= 0\\
x_2 - 3x_3 + x_4 &= 0\\
x_5 &= 1
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
x_1 &= -5t - 2u\\
x_2 &= 3t - u\\
x_3 &= t\\
x_4 &= u\\
x_5 &= 1
\end{aligned}
\quad , t, u \in \mathbb{R}
$$

(8)

2. Consider the system of equations

$$\begin{aligned} -x_1 + 3x_2 + 2bx_3 &= 0 \\ -x_1 + (b+4)\,x_2 + (2b+1)\,x_3 &= -a \\ (b+1)\,x_2 + (b-1)\,x_3 &= -a+3 \\ -x_1 + 3x_2 + (3b-2)\,x_3 &= 3 \end{aligned}$$

i) Find the values of a and b so that the system has
a) no solution
b) a unique solution
c) infinitely many solutions
ii) Find the solution for the special case $a = 1, b = 3$.

$$\left[\begin{array}{ccc|c} -1 & 3 & 2b & 0 \\ -1 & b+4 & 2b+1 & -a \\ 0 & b+1 & b-1 & -a+3 \\ -1 & 3 & 3b-2 & 3 \end{array}\right] \xrightarrow[-R_1+R_4\to R_4]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c} -1 & 3 & 2b & 0 \\ 0 & b+1 & 1 & -a \\ 0 & b+1 & b-1 & -a+3 \\ 0 & 0 & b-2 & 3 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|c} -1 & 3 & 2b & 0 \\ 0 & b+1 & 1 & -a \\ 0 & 0 & b-2 & 3 \\ 0 & 0 & b-2 & 3 \end{array}\right] \xrightarrow{-R_3+R_4\to R_4} \left[\begin{array}{ccc|c} -1 & 3 & 2b & 0 \\ 0 & b+1 & 1 & -a \\ 0 & 0 & b-2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (4)$$

i) a) If $b=2$ then $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 3 \end{array}\right]$ no soln. (4)

$b=-1$ and $a\neq 1$ $\Rightarrow$ no soln.

b) If $b\neq 2$ and $b\neq -1$, $b\in\mathbb{R}$ unique soln. (4)

c) If $b=-1$ then $\left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 3-3a \\ 0 & 0 & 0 & 0 \end{array}\right]$

$b=-1 \quad a=1$ then infinitely many soln's. (4)

ii) If $b=3,\ a=1$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 3 & 6 & 0 \\ 0 & 4 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_3+R_2\to R_2]{-6R_3+R_1\to R_1} \left[\begin{array}{ccc|c} -1 & 3 & 0 & -18 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (4)$$

$$\xrightarrow{\frac{1}{4}R_2\to R_2} \left[\begin{array}{ccc|c} -1 & 3 & 0 & -18 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-3R_2+R_1\to R_1]{-R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$X_1 = 15$

$X_2 = -1$

$X_3 = 3$

3. Given that A and B are 5×5 matrices with $\det(A) = 4$ and $\det(B) = -3$. Find, if possible

a) $\det\left(B^3\right)$
b) $\det(5A)$
c) $\det\left(AB^{-1}\right)$
d) $\det(A+B)$
e) $\det\left(A^{-1}\right)^T$
f) $\det\left(\dfrac{3}{5}B^{-1}\right)$
g) $\det(2A)^{-1}$.

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3 a) $\det(B^3) = \det(B).\det(B).\det(B) = (-3).(-3).(-3) = -27$

3 b) $\det(5A) = 5^5.\det(A) = 5^5.4$

3 c) $\det(A.B^{-1}) = \det(A).\det(B^{-1}) = \det(A).\dfrac{1}{\det(B)} = 4.\dfrac{1}{-3} = -4/3$

2 d) $\det(A+B)$ can not be determined

3 e) $\det(A^{-1})^T = \det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{4}$

3 f) $\det\left(\frac{3}{5}B^{-1}\right) = \dfrac{3^5}{5^5}.\det(B^{-1}) = \dfrac{3^5}{5^5}.\dfrac{1}{\det B} = \dfrac{3^5}{5^5}.\dfrac{1}{(-3)} = -\dfrac{3^4}{5^5}$

3 g) $\det(2A)^{-1} = \dfrac{1}{\det(2A)} = \dfrac{1}{2^5.\det(A)} = \dfrac{1}{2^5.4} = \dfrac{1}{2^7}$

c) $Ax=b$

$$\overset{A}{\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 12 \\ -12 \\ 8 \end{bmatrix}}$$

$$x = A^{-1} b = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ 1/4 & -1/4 & -1/4 \\ 1/6 & 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} 12 \\ -12 \\ 8 \end{bmatrix} = \begin{bmatrix} -6+6+4 \\ 3+3-2 \\ 2-\cancel{6}+\frac{8}{6} \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{bmatrix}$$

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4. Let

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

a) Find A^{-1} by elementary row operations on an identity matrix.
b) Find A^{-1} by cofactor expansion.
c) Solve the system of equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 &= -12 \\ 2x_1 + 2x_2 + 3x_3 &= 8 \end{aligned}$$

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a) $[A:I] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{3R_2+R_3\to R_3}$

$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3+R_2\to R_2]{-\frac{1}{2}R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1/2 & -3/2 & -1/2 \\ 0 & 2 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1\to R_1}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 2 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \xrightarrow[\frac{1}{6}R_3\to R_3]{\frac{1}{2}R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right]$

The right-hand block is A^{-1}.

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b) $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

$\det(A) = 1.(-6-0) - 4(-3-0) + 3(-2+4)$
$\det(A) = -6 + 12 + 6 = 12$

$C_{11} = (-1)^{1+1}.M_{11} = \begin{vmatrix} -2 & 0 \\ 2 & 3 \end{vmatrix} = -6$

$C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = 3$

$C_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} -1 & -2 \\ 2 & 2 \end{vmatrix} = 2$

$C_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix} = 6$

$C_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3$

$C_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} = 6$

$C_{31} = (-1)^{3+1} M_{31} = +\begin{vmatrix} 4 & 3 \\ -2 & 0 \end{vmatrix} = 6$

$C_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = -3$

$C_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & 4 \\ -1 & -2 \end{vmatrix} = 2$

$\mathrm{Cof}(A) = \begin{bmatrix} -6 & 3 & 2 \\ -6 & -3 & 6 \\ 6 & -3 & 2 \end{bmatrix}$, $\mathrm{Adj}(A) = [\mathrm{Cof}(A)]^T$

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$A^{-1} = \frac{1}{\det(A)}.\mathrm{Adj}(A) = \frac{1}{12}\begin{bmatrix} -6 & -6 & 6 \\ 3 & -3 & -3 \\ 2 & 6 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ 1/4 & -1/4 & -1/4 \\ 1/6 & 1/2 & 1/6 \end{bmatrix}$

5. Determine whether the following sets are subspaces or not
a) $W_1 = \{a_0 + a_1x + a_2x^2 + a_3x_3 \in P_3 \mid a_0, a_1, a_2, a_3 \text{ are integers}\}$
b) $W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x = 3y\}$.

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a) $W_1 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3 \mid a_0, a_1, a_2, a_3 \text{ are integers}\}$

zero polynomial $\in W_1 \Rightarrow W_1 \neq \emptyset$

Let $\alpha = a_0 + a_1x + a_2x^2 + a_3x^3 \in W_1$, where $a_0, a_1, a_2, a_3 \in \mathbb{Z}$

$\beta = b_0 + b_1x + b_2x^2 + b_3x^3 \in W_1$, " $b_0, b_1, b_2, b_3 \in \mathbb{Z}$

$$\alpha + \beta = \underbrace{(a_0+b_0)}_{\in\mathbb{Z}} + \underbrace{(a_1+b_1)}_{\in\mathbb{Z}}x + \underbrace{(a_2+b_2)}_{\in\mathbb{Z}}x^2 + \underbrace{(a_3+b_3)}_{\in\mathbb{Z}}x^3 \in W_1$$

Let $\alpha = 1 + x + x^2 + x^3 \in W_1$ and $k = \frac{1}{2} \in \mathbb{R}$

$$k\alpha = \frac{1}{2}\alpha = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{2}x^3 \notin W_1$$

So W_1 is not a subspace of P_3.

10

b) $W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x = 3y\}$

$(0,0,0) \in W_2 \Rightarrow W_2 \neq \emptyset$

Let $\alpha = (x_1, y_1, z_1) \in W_2$ with $2x_1 = 3y_1$

$\beta = (x_2, y_2, z_2) \in W_2$ " $2x_2 = 3y_2$

$\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2) \Rightarrow$

$$2(x_1+x_2) = 2x_1 + 2x_2 = 3y_1 + 3y_2 = 3(y_1+y_2)$$

$$\Rightarrow 2(x_1+x_2) = 3(y_1+y_2) \Rightarrow \alpha + \beta \in W_2$$

$\forall k \in \mathbb{R}$, $\alpha \in W_2 \Rightarrow k\alpha = k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$

$\Rightarrow$ ~~$2kx = 2k\,3y$~~ $2kx_1 = k.2x_1 = k3y_1 = 3ky_1$

$\Rightarrow 2(kx_1) = 3(ky_1) \Rightarrow k\alpha \in W_2$

So, W_2 is a subspace of $\mathbb{R}^3$.