ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

$(\sin 2x)$

MCS 231
Linear Algebra

Final

January 13, 2010
09:00-10:50

$\ln(g(x))$

$\frac{1}{g(x)} \cdot g'(x)$

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

- The exam consists of **6** questions. **SOLVE 5 QUESTIONS.**
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
25	25	25	20	20	20	

d) $W \leftrightarrow$ row space of A 1

6

$W^{\perp} \leftrightarrow$ null space of A

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ 1 } \longrightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \text{ 1}$$

$x_3 = t$, $x_4 = s$

$x_1 = t + 2s$

$x_2 = -s$

$x_3 = t$

$x_4 = s$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t+2s \\ -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} s \text{ 2 } , t, s \in \mathbb{R}$$

$V = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ 1 form a basis for $W^{\perp}$.

1. Let $W = \text{Span}\,\{\overset{\beta_1}{(-1,0,1,2)}, \overset{\beta_2}{(0,1,0,1)}\}$ and let $\alpha = (-1,2,6,0) \in \mathbb{R}^4$.
a) Find an orthonormal basis for W.
b) Find $\text{proj}_W\, \alpha$ and $d(\alpha, W)$.
c) Write $\alpha = \beta_1 + \beta_2$ where $\beta_1 \in W$ and $\beta \in W^\perp$.
d) Find a basis for the orthogonal complement of W.

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a) $\alpha_1 = \beta_1 = (-1,0,1,2)$ (1)

$\alpha_2 = \beta_2 - \dfrac{\langle \beta_2, \alpha_1\rangle}{\|\alpha_1\|^2}\,\alpha_1$ (1)

$\alpha_2 = (0,1,0,1) - \dfrac{\langle (0,1,0,1),(-1,0,1,2)\rangle}{6}(-1,0,1,2)$ (1)

$\alpha_2 = (0,1,0,1) - \dfrac{1}{3}(-1,0,1,2) = (1/3, 1, -1/3, 1/3)$ (1)

8// $\alpha_1 = (-1,0,1,2)$ and $\alpha_2 = (1/3,1,-1/3,1/3)$ are orthogonal basis for W. $\|\alpha_1\| = \sqrt{6}$ (1) $\|\alpha_2\| = \sqrt{\frac{1}{9}+1+\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{1}{3}+1} = \frac{2}{\sqrt{3}}$ (1)

So $\gamma_1 = \dfrac{\alpha_1}{\|\alpha_1\|} = \dfrac{(-1,0,1,2)}{\sqrt{6}} = \left(-\frac{1}{\sqrt6}, 0, \frac{1}{\sqrt6}, \frac{2}{\sqrt6}\right) = \frac{1}{\sqrt6}(-1,0,1,2)$ (1)

$\gamma_2 = \dfrac{\alpha_2}{\|\alpha_2\|} = \dfrac{(1/3,1,-1/3,1/3)}{\frac{2}{\sqrt3}} = \dfrac{(1/3,1,-1/3,1/3)}{\frac{2\sqrt3}{3}} = \dfrac{1}{2\sqrt3}(1,3,-1,1)$ (1) $\frac{\sqrt3}{6}$

are orthonormal basis for W.

b) $\text{Proj}_W^\alpha = \langle \alpha, \gamma_1\rangle \gamma_1 + \langle \alpha, \gamma_2\rangle \gamma_2$ (2)

$\text{Proj}_W^\alpha = \dfrac{\langle(-1,2,6,0),(-1,0,1,2)\rangle}{6}(-1,0,1,2) + \dfrac{\langle(-1,2,6,0),(1,3,-1,1)\rangle}{12}(1,3,-1,1)$ (1)

4// $\text{Proj}_W^\alpha = \frac{7}{6}(-1,0,1,2) + \frac{1}{12}(1,3,-1,1) = (-5/4, -1/4, 5/4, 9/4)$ (1)

$d(\alpha, W) = \|\alpha - \text{Proj}_W^\alpha\| = \|(-1,2,6,0) - (-5/4,-1/4,5/4,9/4)\|$ (1)

3// $d(\alpha, W) = \|(1/4, 9/4, 19/4, -9/4)\|$ (1) $= \sqrt{\frac{1}{16}+\frac{81}{16}+\frac{361}{16}+\frac{81}{16}} = \dfrac{\sqrt{524}}{4}$ (1)

c) $\alpha = \beta_1 + \beta_2$, $\beta_1 \in W$, $\beta_2 \in W^\perp$

$\beta_1 = \text{Proj}_W^\alpha = (-5/4, -1/4, 5/4, 9/4) \in W$ (1)

4// $\beta_2 = \alpha - \beta_1 = (-1,2,6,0) - (-5/4,-1/4,5/4,9/4)$ (1)

$\beta_2 = (1/4, 9/4, 19/4, -9/4) \in W^\perp$. (2)

$$q_3 = \frac{P_3}{\|P_3\|} = \frac{(1, 1/2, 1)}{\sqrt{\frac{9}{4}}} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \quad (1)$$

$$Q = [q_1 \mid q_2 \mid q_3] = \begin{bmatrix} 1/\sqrt{5} & \frac{-4}{3\sqrt{5}} & 2/3 \\ 2/\sqrt{5} & -2/3\sqrt{5} & 1/3 \\ 0 & 5/3\sqrt{5} & 2/3 \end{bmatrix} \quad (1)$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad (1)$$

b) $\lambda_1 = \lambda_2 = -1 \rightarrow P_1 = (-1/2, 1, 0)$, $\rightarrow P_2 = (-1, 0, 1)$

$\lambda_1' = \lambda_2' = (-1)^{13} = -1$ (1) eigenvalues of A^{13} and P_1, P_2 are eigenvectors of A^{13} corr. to $\lambda_1' = \lambda_2' = -1$ (1)

$\lambda_3 = 8 \rightarrow P_3 = (1, 1/2, 1) \Rightarrow \lambda_3' = (8)^{13}$ (1) is eigenvalue of A^{13} and P_3 is eigenvector corr to $\lambda_3' = 8^{13}$ (1)

2. Let

$$A = \begin{bmatrix} 3 & a & 4 \\ 2 & 0 & 2 \\ b & 2 & 3 \end{bmatrix}.$$

a) Find real number a, b, an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.
b) Write the eigenvalues and the eigenvectors of A^{13}.

---

a) $A^T = A$. (A is symmetric) $\Rightarrow a = 2, b = 4$ (2)

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \quad \det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda-3 \end{vmatrix} \quad (2)$$

$$\det(\lambda I - A) = (\lambda-3)(\lambda^2 - 3\lambda - 4) + 2(-2\lambda-2) - 4(4+4\lambda)$$

$$= (\lambda+1)^2(\lambda-8) = 0$$

$\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 8$ are eigenvalues of A. (3)

For $\lambda_1 = \lambda_2 = -1 \Rightarrow \begin{bmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_2 = t,\ x_3 = s,\ x_1 = -\frac{1}{2}t - s$

$$X = \underbrace{\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}_{P_1} t + \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{P_2} s, \quad t, s \in \mathbb{R} \quad (2)$$

For $\lambda_3 = 8 \Rightarrow \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_3 = t,\ x_1 = t,\ x_2 = \frac{1}{2}t$ $\quad X = \underbrace{\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}}_{P_3} t, \ t \in \mathbb{R}$ (1)

$\langle P_1, P_2 \rangle = \langle (-1/2, 1, 0), (-1, 0, 1) \rangle = \frac{1}{2} \neq 0$ not orthogonal (1)

$\langle P_1, P_3 \rangle = \langle (-1/2, 1, 0), (1, 1/2, 1) \rangle = 0$ orth. (1)

$\langle P_2, P_3 \rangle = \langle (-1, 0, 1), (1, 1/2, 1) \rangle = 0$ orth.

use G.S.O.P

$P_1 = (-1/2, 1, 0)$, $P_2 = (-1, 0, 1)$

$\alpha_1 = P_1 = (-1/2, 1, 0)$ (1)

$$\alpha_2 = P_2 - \frac{\langle P_2, \alpha_1 \rangle}{\|\alpha_1\|^2} \alpha_1 \ (1) = (-1, 0, 1) - \frac{\langle (-1,0,1), (-1/2, 1, 0) \rangle}{\frac{5}{4}} (-1/2, 1, 0) \ (1)$$

$$\alpha_2 = (-1, 0, 1) - \frac{1/2}{5/4}(-1/2, 1, 0) = \left(-\frac{4}{5}, -\frac{2}{5}, 1\right) \ (1)$$

α_1, α_2 and P_3 are orthogonal.

$$q_1 = \frac{\alpha_1}{\|\alpha_1\|} = \frac{(-1/2, 1, 0)}{\frac{\sqrt{5}}{2}} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \ (1)$$

$$q_2 = \frac{\alpha_2}{\|\alpha_2\|} = \frac{(-4/5, -2/5, 1)}{\sqrt{\frac{9}{5}}} = \frac{1}{3\sqrt{5}}(-4, -2, 5) \ (1)$$

3. Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear operator defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ x_1 + 5x_3 \\ x_1 + 3x_2 - 2x_3 \end{bmatrix}.$$

a) Find the representation matrix A of L with respect to the standard basis S for $\mathbb{R}^3$.
b) Find the representation matrix B of L with respect to the basis $T = \{\underset{V_1}{(1,1,0)}, \underset{V_2}{(3,-2,1)}, \underset{V_3}{(3,-1,1)}\}$.
c) Verify that A and B are similar.

---

a) $S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

$A = \left[[L(e_1)]_S \mid [L(e_2)]_S \mid [L(e_3)]_S \right]$

$L(e_1) = L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\1\end{bmatrix}$ $\quad L(e_2) = L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\3\end{bmatrix}$ $\quad L(e_3) = L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\5\\-2\end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 5 \\ 1 & 3 & -2 \end{bmatrix}$ 6

b) $B = \left[[L(V_1)]_T \mid [L(V_2)]_T \mid [L(V_3)]_T \right]$ 1

$L(V_1) = L\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\4\end{bmatrix}$ 1, $\quad L(V_2) = L\left(\begin{bmatrix}3\\-2\\1\end{bmatrix}\right) = \begin{bmatrix}9\\8\\-5\end{bmatrix}$ 1, $\quad L(V_3) = L\left(\begin{bmatrix}3\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}8\\8\\-2\end{bmatrix}$ 1

$$\begin{array}{cccccc} V_1 & V_2 & V_3 & L(V_1) & L(V_2) & L(V_3) \end{array}$$
$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 3 & 1 & 9 & 8 \\ 1 & -2 & -1 & 1 & 8 & 8 \\ 0 & 1 & 1 & 4 & -5 & -2 \end{array}\right] \text{1} \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 3 & 1 & 9 & 8 \\ 0 & -5 & -4 & 0 & -1 & 0 \\ 0 & 1 & 1 & 4 & -5 & -2 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{5R_3+R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 3 & 1 & 9 & 8 \\ 0 & 1 & 1 & 4 & -5 & -2 \\ 0 & 0 & 1 & 20 & -26 & -10 \end{array}\right] \xrightarrow[-R_3+R_2 \to R_2]{-3R_3+R_1 \to R_1} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 0 & -59 & 87 & 38 \\ 0 & 1 & 0 & -16 & 21 & 8 \\ 0 & 0 & 1 & 20 & -26 & -10 \end{array}\right] \xrightarrow{-3R_2+R_1 \to R_1} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -11 & 24 & 14 \\ 0 & 1 & 0 & -16 & 21 & 8 \\ 0 & 0 & 1 & 20 & -26 & -10 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{B}$ 4

c) We must show that $B = P^{-1}AP$ 1 where P is the transition matrix from T to S.

$P = \left[[V_1]_S \mid [V_2]_S \mid [V_3]_S \right] = \begin{bmatrix} 1 & 3 & 3 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ 1

$$\begin{array}{cc} P & I \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{1} \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -5 & -4 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{5R_3+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[-R_3+R_2 \to R_2]{-3R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & -3 & -15 \\ 0 & 1 & 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{-3R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & -1 & 1 & 5 \end{array}\right]$$

$\underbrace{\qquad}_{I} \quad \underbrace{\qquad}_{P^{-1}}$ 4

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & -3 \\ 1 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 5 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3 \\ 1 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 9 & 8 \\ 1 & 8 & 8 \\ 4 & -5 & -2 \end{bmatrix} = \begin{bmatrix} -11 & 24 & 14 \\ -16 & 21 & 8 \\ 20 & -26 & -10 \end{bmatrix} = B$$

2 1

So, B and A are similar

$$P_1 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} \qquad \|P_1\| = 5/3$$

$$P_2 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \qquad \|P_2\| = \frac{5}{4}$$

$$q_1 = \frac{P_1}{\|P_1\|} =$$

4. Find the standard equation of the quadratic equation and classify the conic section

$$9x_1^2 + 24x_1x_2 + 16x_2^2 - 20x_1 + 15x_2 = 0$$

$a=9$, $b=12$ $c=16$ $d=-20$ $e=15$ $f=0$ (2)

$$[x_1 \ \ x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [d \ \ e]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\underset{X^T}{[x_1 \ \ x_2]}\underset{A}{\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} + [-20 \ \ 15]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (1)$$

$$A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad \det(\lambda I - A) \underset{(1)}{=} \begin{vmatrix} \lambda-9 & -12 \\ -12 & \lambda-16 \end{vmatrix} = (\lambda-9)(\lambda-16) - 144 = 0$$

$$= \lambda^2 - 25\lambda + 144 - 144 = 0 \quad (2)$$

$$= \lambda^2 - 25\lambda = \lambda(\lambda - 25) = 0 \quad \lambda_1 = 0, \ \lambda_2 = 25$$

For $\lambda_1 = 0 \Rightarrow \begin{bmatrix} -9 & -12 \\ -12 & -16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4/3 \\ 1 & 4/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}$

$x_2 = t$ $\quad X = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} t, \ t \in \mathbb{R} \quad P_1 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ (2) $\quad \|P_1\| = \frac{5}{3}$ (1)

$x_1 = -\frac{4}{3}t$

For $\lambda_2 = 25 \Rightarrow \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{3}{4} \\ -1 & \frac{3}{4} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}$ $\quad x_2 = t$, $x_1 = \frac{3}{4}t$

$X = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} t, \ t \in \mathbb{R}$

$P_2 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ (2) $\quad \|P_2\| = 5/4$ (1)

$\cancel{q_1 = \frac{P_1}{\|P_1\|} = \frac{5}{3}\begin{bmatrix} -4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 5/3 \end{bmatrix}}$ $\quad \cancel{q_1 = \frac{P_1}{\|P_1\|} = \frac{5}{3}\begin{bmatrix} -4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -20/9 \\ 5/3 \end{bmatrix}}$

$\cancel{q_2 = \frac{P_2}{\|P_2\|} = \frac{5}{4}}$ $\quad \cancel{q_2 = \frac{P_2}{\|P_2\|} = \frac{5}{4}\begin{bmatrix} 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15/16 \\ 5/4 \end{bmatrix}}$

$q_1 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$ $\quad Q = [q_1 | q_2]$

$q_2 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ (1)

$\cancel{Q = [q_1 | q_2] = \begin{bmatrix} -20/9 & 15/16 \\ 5/3 & 5/4 \end{bmatrix}}$ $\quad X = QY$ change variables

(1) $\quad \cancel{=0 \Rightarrow [y_1 \ y_2]\begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + [-20 \ \ 15]\begin{bmatrix} -20/9 & 15/16 \\ 5/3 & 5/4 \end{bmatrix}}$ (1)

$$Y^T D Y + [-20 \ \ 15]\, QY = 0$$

$$[y_1 \ y_2]\begin{bmatrix} 0 \\ 25y_2 \end{bmatrix} + [-20 \ \ 15]\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \quad (2)$$

$$25y_2^2 + [25 \ \ 0]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \quad (1)$$

$$25y_2^2 + 25y_1 = 0 \Rightarrow \boxed{y_2^2 + y_1 = 0} \quad \text{parabola} \ (1)$$

(1)

5. Let

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}.$$

a) Show that A is Hermitian.
b) Find a unitary matrix U that diagonalizes A.

---

a) If $A^* = A$ 1 then A is Hermitian.

6 $\bar{A} = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} \Rightarrow (\bar{A})^T = A^* = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} = A \Rightarrow A$ is Hermitian. 1 1 2

b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -1-i \\ -1+i & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3)-2 = \lambda^2-5\lambda+4 = (\lambda-1)(\lambda-4) = 0$ 1 1

$\lambda_1 = 1$, $\lambda_2 = 4$ are eigenvalues of A 2

For $\lambda_1 = 1 \Rightarrow \begin{bmatrix} -1 & -1-i \\ -1+i & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1+i \\ -1+i & -2 \end{bmatrix} \xrightarrow{(1-i)R_1} \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$

$x_2 = s$
$x_1 = (-1-i)s$ $\quad X = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} s$ 2

~~$x_2 = s$~~

$P = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$, 2 $\quad \|P_1\| = \sqrt{3}$ 1 $\quad u_1 = \dfrac{P_1}{\|P_1\|} = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ 1 corr. to $\lambda_1 = 1$

For $\lambda_2 = 4 \Rightarrow \begin{bmatrix} 2 & -1-i \\ -1+i & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{-1-i}{2} \\ -1+i & 1 \end{bmatrix} \xrightarrow{(1-i)R_1} \begin{bmatrix} 1 & \frac{-1-i}{2} \\ 0 & 0 \end{bmatrix}$

$x_1 = \left(\frac{1+i}{2}\right)s$
$x_2 = s$ $\quad \Rightarrow X = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix} s$, 2 $\quad P_2 = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$ $\quad \|P_2\| = \frac{\sqrt{6}}{2}$ 1

$u_2 = \dfrac{P_2}{\|P_2\|} = \begin{bmatrix} \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ 1

$U = [u_1 \,\vdots\, u_2] = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$ 1 diagonalizes A and

$U^{-1}AU = U^*AU = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ 1 $\quad$ ~~or~~ $U^{-1} = U^*$

6. Prove that
a) if A and B are similar matrices, then $\det(A) = \det(B)$.
b) if Q is an orthogonal matrix, then $\det(Q) = \pm 1$.

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a) If A and B are similar matrices; ~~A~~then there exists an invertible matrix P such that

$A = P^{-1}BP$ ③ $\Rightarrow$ $\det(A) = \det(P^{-1}BP)$ ④

$\det(A) = \det(P^{-1}) \cdot \det(B) \cdot \det(P)$ ③

$\det(A) = \frac{1}{\cancel{\det P}} \cdot \det(B) \cdot \cancel{\det(P)}$ ④ (P is invertible) ($\det(P) \neq 0$)

$\Rightarrow \det(A) = \det(B)$

b) If Q is an orthogonal matrix then

$Q^{-1} = Q^T \Rightarrow$ ~~det~~ $Q \cdot Q^{-1} = I$

$\Rightarrow Q \cdot Q^T = I \Rightarrow \det(Q \cdot Q^T) = \det(I)$

$\det Q \cdot \det Q^T = 1$ (since $\det Q^T = \det Q$)

$\Rightarrow (\det Q)^2 = 1 \Rightarrow \det Q = \mp 1.$