ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MCS 231 Linear Algebra

1st Midterm
November 10, 2010
08:40-10:30

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 5 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	24	104

Question 1. Consider the linear system

$$\begin{aligned} 4x_1 - 3x_2 - 11x_3 + 4x_4 - 6x_5 &= -9 \\ -2x_1 + 2x_2 + 6x_3 - 6x_5 &= -6 \\ 3x_1 - 2x_2 - 8x_3 + 3x_4 - 5x_5 &= -7 \\ x_1 - x_2 - 3x_3 + x_4 - x_5 &= -2 \\ -x_1 + 2x_2 + 4x_3 + 3x_4 - 17x_5 &= -19. \end{aligned}$$

(a) Form the augmented matrix of the given system.

(b) Find the reduced row echelon form of the augmented matrix, which is found in part (a).

(c) Solve the system if it is consistent.

Answer 1.

(a) The augmented matrix of the given system is

$$\left[\begin{array}{rrrrr|r} 4 & -3 & -11 & 4 & -6 & -9 \\ -2 & 2 & 6 & 0 & -6 & -6 \\ 3 & -2 & -8 & 3 & -5 & -7 \\ 1 & -1 & -3 & 1 & -1 & -2 \\ -1 & 2 & 4 & 3 & -17 & -19 \end{array}\right].$$

(b) It is straightforward to show that

$$\left[\begin{array}{rrrrr|r} 4 & -3 & -11 & 4 & -6 & -9 \\ -2 & 2 & 6 & 0 & -6 & -6 \\ 3 & -2 & -8 & 3 & -5 & -7 \\ 1 & -1 & -3 & 1 & -1 & -2 \\ -1 & 2 & 4 & 3 & -17 & -19 \end{array}\right] \overset{\text{rref}}{\rightsquigarrow} \left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(c) Using part (b) and putting $x_3 = s$, $x_5 = t$, we get $x_1 = 2 + 2s - t$, $x_2 = -1 - s + 2t$, and $x_4 = -5 + 4t$. Therefore the solution set is

$$x_1 = 2 + 2s - t,\ x_2 = -1 - s + 2t,\ x_3 = s,\ x_4 = -5 + 4t,\ x_5 = t \quad \text{where} \quad s, t \in \mathbb{R}.$$

Question 2. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \\ 4 & 3 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

(a) Find the LU decomposition of A.
(b) Use this decomposition to solve $Ax = b$.

Answer 2.

a) $$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 2 & 1 & 0 \\ 0 & 7 & -9 & -4 & 0 & 1 \end{array}\right]$$

$l_{21} = -2$
$l_{31} = 4$

$$\xrightarrow{7R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 2 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 19}_{U} & & & \underbrace{10 \quad 7 \quad 1}_{L^{-1}} & & \end{array}\right] \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$$

$l_{32} = -7$

$$Ax = (LU)x = L(Ux) = b \text{ and } Ux = Z \qquad LZ = b$$

$$LZ = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

$$z_1 = 4, \quad \left.\begin{array}{l} -2z_1 + z_2 = -1 \\ 4z_1 - 7z_2 + z_3 = 2 \end{array}\right\} \quad \begin{array}{l} z_2 = 7 \\ z_3 = 35 \end{array}$$

$$\Rightarrow Z = \begin{bmatrix} 4 \\ 7 \\ 35 \end{bmatrix} \text{ and } Ux = Z \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 35 \end{bmatrix}$$

$$19x_3 = 35 \Rightarrow x_3 = \frac{35}{19}$$

$$-x_2 + 4x_3 = 7 \Rightarrow x_2 = \frac{7}{19}$$

$$x_1 - x_2 + 2x_3 = 4 \Rightarrow x_1 = \frac{13}{19}$$

Question 3. Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

(a) Find $|A|$.
(b) Find $|\operatorname{adj}(A)|$.

Answer 3.

(a) *Method 1*: Using elementary row operations, simplify the calculations:

$$\begin{vmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -4 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -8.$$

Method 2: Apply cofactor expansion along any row, e.g. the first row:

$$|A| = \begin{vmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Note that the second and the third determinants on the right hand side are 0, since they have identical rows. Hence,

$$|A| = \begin{vmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}. \tag{1}$$

Let us continue to evaluate sub-determinants:

$$\begin{vmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2 + 0 - 2 = -4$$

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 2 + 2 - 0 = 4.$$

Substituting these results in (1), we get $|A| = -4 - 4 = -8$.

(b) Since $A \cdot \operatorname{adj}(A) = |A|I$, where I is 4×4 identity matrix, we have $A \cdot \operatorname{adj}(A) = -8I$, and therefore

$$|A \cdot \operatorname{adj}(A)| = (-8)^4 |I| = (-8)^4.$$

On the other hand, it is known that

$$|A \cdot \operatorname{adj}(A)| = |A||\operatorname{adj}(A)| = -8|\operatorname{adj}(A)|.$$

Combining these two equations, we obtain

$$-8|\operatorname{adj}(A)| = (-8)^4,$$

and therefore

$$|\operatorname{adj}(A)| = (-8)^3 = -512.$$

Alternatively, you can construct $\operatorname{adj}(A)$ and evaluate $|\operatorname{adj}(A)|$ directly!

Question 4. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 1 & 4 & 7 & 6 \end{bmatrix}$. Find

(a) $|A|$,

(b) $|D|$ if $A \quad \overset{2R_1+R_2\to R_2}{\longmapsto} \quad B \quad \overset{3R_3\to R_3}{\longmapsto} \quad C \quad \overset{R_1\leftrightarrow R_4}{\longmapsto} \quad D$,

(c) $|AC^{-1}|$,

(d) $|A^5|$,

(e) $|(B^T)^{-1}|$,

(f) $|2D|$,

(g) $\left|\left(\frac{2}{3}A\right)^{-1}\right|$,

(h) $\left|\frac{5}{4}C^{-2}\right|$.

Answer 4.

(a) By using the elementary row operations, it easy to see that

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 1 & 4 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -7 & -11 \\ 0 & 2 & 4 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & -1 & -3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -10.$$

(b) $|A| = |B| = -10$, $|C| = 3\,|B| = -30$ and $|D| = -|C| = 30$.

(c) $|AC^{-1}| = \dfrac{|A|}{|C|} = \dfrac{1}{3}$.

(d) $|A^5| = |A|^5 = -10^5 = -100000$.

(e) $|(B^T)^{-1}| = \dfrac{1}{|B^T|} = \dfrac{1}{|B|} = -\dfrac{1}{10}$.

(f) $|2D| = 2^4|D| = 480$.

(g) $\left|\left(\dfrac{2}{3}A\right)^{-1}\right| = \dfrac{1}{|\frac{2}{3}A|} = \dfrac{1}{(\frac{2}{3})^4|A|} = -\dfrac{81}{160}$.

(h) $\left|\dfrac{5}{4}C^{-2}\right| = \left(\dfrac{5}{4}\right)^4 |C^{-2}| = \left(\dfrac{5}{4}\right)^4 |C^{-1}|^2 = \left(\dfrac{5}{4}\right)^4 \dfrac{1}{|C|^2} = \dfrac{625}{3600}$.

Question 5.

(a) Let A and B are $n \times n$ matrices. Show that $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.
(b) Show that if A is invertible then it has a unique inverse.
(c) Let A and B are nonsingular $n \times n$ matrices. Show that AB is also nonsingular and moreover $(AB)^{-1} = B^{-1}A^{-1}$.
(d) Let A and B are $n \times n$ symmetric matrices satisfying $AB = BA$. Show that AB is also symmetric.

Answer 5.

a) $tr(A+B) = (A+B)_{11} + (A+B)_{22} + (A+B)_{33} + \cdots + (A+B)_{nn}$

$tr(A+B) = (A_{11}+B_{11}) + (A_{22}+B_{22}) + (A_{33}+B_{33}) + \cdots + (A_{nn}+B_{nn})$

$tr(A+B) = (A_{11}+A_{22}+A_{33}\cdots+A_{nn}) + (B_{11}+B_{22}+B_{33}+\cdots+B_{nn})$

$tr(A+B) = tr(A)+tr(B)$

b) If A is invertible with $AB=AC$ then we must show that $B=C$, We've $AB=BA=I$ and $AC=CA=I$,

$$B = BI = B(AC) = \underbrace{(BA)}_{I}.C = IC = C$$

c) A and B are nonsiggular, $det(A) \neq 0$ and $det(B) \neq 0$. Since $det(A.B) = det(A).det(B) \neq 0 \Rightarrow AB$ is nonsingular ~~and~~

~~$(AB)(B^{-1}A^{-1}$~~

And we must show that $(AB).(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})AB=I$

$$(AB)(B^{-1}A^{-1}) = A\underbrace{B.B^{-1}}_{I}.A^{-1} = A.A^{-1} = I$$

and $$(B^{-1}.A^{-1})(AB) = B^{-1}\underbrace{A^{-1}.A}_{I}.B = B^{-1}B = I$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

d) We must show that $(AB)^T = AB$

We know that $(AB)^T = B^TA^T$ and A,B are symmetric matrices, that is, $A^T=A$ and $B^T=B$

$\Rightarrow (AB)^T = B^TA^T = BA$, since it is given that $BA=AB$

$\Rightarrow (AB)^T = AB$, AB is symmetric.

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MCS 231 Linear Algebra

2$^{\text{nd}}$ Midterm
December 24, 2010
10:40-12:30

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit.
- Calculators are <u>not</u> allowed.

GOOD LUCK!

Please do <u>not</u> write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
10	25	25	10	15	25	110

Question 1. In each part, determine whether W is a subspace or not. Justify your answer.

(a) $W = \left\{ \begin{bmatrix} a & a+b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad \text{where} \quad c < 0 \right\}.$

(b) $W = \{(x, y, z) \in \mathbb{R}^3 \quad \text{where} \quad 2x + 5z = 0\}.$

Answer 1.

(a) Not a subspace.

(b) Subspace.

Question 2. Let

$$A = \begin{bmatrix} 1 & -3 & -1 & 2 & 2 \\ 2 & -5 & 2 & 1 & 0 \\ 3 & -5 & 13 & -6 & -9 \\ -1 & -1 & -15 & 10 & 13 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Find a vector form of the general solution of the system $Ax = b$.
(b) Find a basis for the row space of A.
(c) Find a basis for the column space of A.
(d) Find a basis for the null space of A.
(e) Find the rank and the nullity of A.

Answer 2.

(a) Note that

$$\left[\begin{array}{rrrrr|r} 1 & -3 & -1 & 2 & 2 & 2 \\ 2 & -5 & 2 & 1 & 0 & 3 \\ 3 & -5 & 13 & -6 & -9 & 0 \\ -1 & -1 & -15 & 10 & 13 & 4 \end{array}\right] \overset{\text{rref}}{\rightsquigarrow} \left[\begin{array}{rrrrr|r} 1 & 0 & 11 & -7 & 0 & -21 \\ 0 & 1 & 4 & -3 & 0 & -9 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Putting $x_3 = s$, $x_4 = t$, we obtain $x_1 = -21 - 11s + 7t$, $x_2 = -9 - 4s + 3t$, and $x_5 = -2$. Therefore a vector form of the general solution of the system $Ax = b$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -21 \\ -9 \\ 0 \\ 0 \\ -2 \end{bmatrix} + s \begin{bmatrix} -11 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) It is clear from part (a) that

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 13 \\ -6 \\ -9 \end{bmatrix} \right\} \quad \text{is a basis for the row space of } A.$$

(c) It is clear from part (a) that

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -5 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -9 \\ 13 \end{bmatrix} \right\} \quad \text{is a basis for the column space of } A.$$

(d) It is clear from part (a) that

$$\left\{ \begin{bmatrix} -11 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{is a basis for the null space of } A.$$

(e) It is clear from part (a) that the rank of A is 3 and the nullity of A is 2.

Question 3. Let

$$\begin{aligned} p_1 &= 1 - 3x + x^2 + 7x^3, \\ p_2 &= -1 - 3x + 5x^2 - x^3, \\ p_3 &= 1 - 2x^2 + 4x^3, \\ p_4 &= 2 - 3x - x^2 + 11x^3. \end{aligned}$$

(a) Find a basis S for the linear span of $\{p_1, p_2, p_3, p_4\}$.

(b) Express each vector p_i not in S as a linear combination of the vectors in S.

(c) Find a basis T for P_3 containing S.

Answer 3.

(a) Note that

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -3 & -3 & 0 & -3 \\ 1 & 5 & -2 & -1 \\ 7 & -1 & 4 & 11 \end{bmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $S = \{p_1,\, p_2\}$ form a basis for the linear span of $\{p_1, p_2, p_3, p_4\}$.

(b) It is clear from part (a) that $p_3 = \dfrac{1}{2}p_1 - \dfrac{1}{2}p_2$ and $p_4 = \frac{3}{2}p_1 - \frac{1}{2}p_2$.

(c) Note that

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -3 & -3 & 0 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ 7 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\text{ref}}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & -1/6 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 2/3 & 1/3 \end{bmatrix}.$$

Therefore $T = \{p_1,\, p_2, 1, x\}$ form a basis for P_3.

Question 4.

(a) Let $L:\ \mathbb{R}^3 \to \mathbb{R}$ such that

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = a_1\, a_2 + a_3.$$

Determine whether L is a linear transformation or not.

(b) Let $L:\ \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 - a_2 \\ 3a_1 \end{bmatrix}.$$

Determine whether L is a linear transformation or not.

Answer 4.

a) $\alpha = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \ \beta = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$L(\alpha+\beta) = L\left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}\right) = (a_1+b_1)(a_2+b_2) + (a_3+b_3)$$

$$= a_1.a_2 + \underbrace{a_1.b_2 + a_2.b_1} + b_1.b_2 + (a_3+b_3)$$

$$L(\alpha) + L(\beta) = a_1.a_2 + a_3 + b_1.b_2 + b_3 \neq L(\alpha+\beta)$$

L is not a linear transformation.

b) $L(\alpha+\beta) = L(\alpha) + L(\beta)$

$L(c\alpha) = cL(\alpha)$, $c \in \mathbb{R}$

L is a linear transformation.

Question 5. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$L\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\1\\0\end{bmatrix},\quad L\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}0\\-1\\1\end{bmatrix}.$$

(a) Find $L\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)$. (b) Find $L\left(\begin{bmatrix}3\\-7\end{bmatrix}\right)$.

Answer 5.

a) $$\begin{bmatrix}a_1\\a_2\end{bmatrix}=c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1&1&a_1\\1&-1&a_2\end{array}\right]\longrightarrow\left[\begin{array}{cc|c}1&0&\overset{c_1}{\overset{\parallel}{\frac{a_1+a_2}{2}}}\\0&1&\underset{c_2}{\underset{\parallel}{\frac{a_1-a_2}{2}}}\end{array}\right]$$

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\left(\frac{a_1+a_2}{2}\right)\begin{bmatrix}1\\1\end{bmatrix}+\left(\frac{a_1-a_2}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$L\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=\left(\frac{a_1+a_2}{2}\right)L\left(\begin{bmatrix}1\\1\end{bmatrix}\right)+\left(\frac{a_1-a_2}{2}\right)\cdot L\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$L\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=\left(\frac{a_1+a_2}{2}\right)\begin{bmatrix}2\\1\\0\end{bmatrix}+\left(\frac{a_1-a_2}{2}\right)\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$L\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=\begin{bmatrix}a_1+a_2\\a_2\\\frac{a_1-a_2}{2}\end{bmatrix}$$

b) $$L\left(\begin{bmatrix}3\\-7\end{bmatrix}\right)=\begin{bmatrix}3-7\\-7\\\frac{3-(-7)}{2}\end{bmatrix}=\begin{bmatrix}-4\\-7\\5\end{bmatrix}$$

Question 6. Let $L: P_1 \to P_1$ be a linear operator defined by

$$L(a_0 + a_1 x) = a_0 + a_1(x+1)$$

and $S = \{\overset{v_1}{6+3x}, \overset{v_2}{10+2x}\}$ and $T = \{2, 3+2x\}$ be two ordered basis for P_1.

(a) Find the representation matrix A of L with respect to S.
(b) Find the representation matrix B of L with respect to T.
(c) Verify that A and B are similar.

Answer 6. $S = \{v_1 = 6+3x, v_2 = 10+2x\}$, $T = \{w_1 = 2, w_2 = 3+2x\}$

a) $A = \left[[L(v_1)]_S \mid [L(v_2)]_S \right]$

$L(v_1) = L(6+3x) = 6+3(x+1) = 9+3x$
$L(v_2) = L(10+2x) = 10+2(x+1) = 12+2x$

$$\begin{array}{cc|c|c} v_1 & v_2 & L(v_1) & L(v_2) \\ 6 & 10 & 9 & 12 \\ 3 & 2 & 3 & 2 \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2/3 & -2/9 \\ 0 & 1 & 1/2 & 4/3 \end{array}\right], \quad A = \begin{bmatrix} 2/3 & -2/9 \\ 1/2 & 4/3 \end{bmatrix}$$

b) $B = \left[[L(w_1)]_T \,\vdots\, [L(w_2)]_T \right]$

$L(w_1) = L(2) = 2$
$L(w_2) = L(3+2x) = 3+2(x+1) = 5+2x$

$$\begin{array}{cc|c|c} w_1 & w_2 & L(w_1) & L(w_2) \\ 2 & 3 & 2 & 5 \\ 0 & 2 & 0 & 2 \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right], \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

c) $B = P^{-1}AP$, where P is the transition matrix T to S.

$$\begin{array}{cc|c|c} v_1 & v_2 & w_1 & w_2 \\ 6 & 10 & 2 & 3 \\ 3 & 2 & 0 & 2 \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2/9 & 7/9 \\ 0 & 1 & 1/3 & -1/6 \end{array}\right], \quad P = \begin{bmatrix} -2/9 & 7/9 \\ 1/3 & -1/6 \end{bmatrix}$$

and $P^{-1} = \begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -2/9 \\ 1/2 & 4/3 \end{bmatrix} \begin{bmatrix} -2/9 & 7/9 \\ 1/3 & -1/6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$$

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MCS 231 - Linear Algebra

FINAL EXAMINATION

17.01.2011

Question	Grade	Out of
1		25
2		20
3		20
4		25
5		15
6		15
Total		120

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: Aynur Baki Gürsoy

DURATION: 120 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Let

$$p_1 = 5 + x - 3x^2 \qquad q_1 = 4 - 3x + x^2$$
$$p_2 = 7 - 2x + 4x^2 \qquad q_2 = 7 - 5x + 2x^2$$
$$p_3 = 1 + x^2 \qquad q_3 = 1 + x + x^2$$

Given that $S = \{p_1, p_2, p_3\}$ and $T = \{q_1, q_2, q_3\}$ are both bases for P_2, perform the following computations:

(a) Find the polynomial p if $[p]_S = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$.

(b) Find the transition matrix $P_{T \leftarrow S}$ from the S-basis to the T-basis.

Let $q = 2 - 4x + 15x^2$.

(c) Find $[q]_S$.

(d) Find $[q]_T$ by using part (b) and (c).

(e) Find $[q]_T$ directly.

Answer 1.

a) $p = -5p_1 + 3p_2 + p_3 = -5(5 + x - 3x^2) + 3(7 - 2x + 4x^2) + (1 + x^2)$

$p = -3 - 11x + 28x^2$

b) $$\left[\begin{array}{ccc|ccc} 4 & 7 & 1 & 5 & 7 & 1 \\ -3 & -5 & 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & -3 & 4 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 76 & -9 & -5 \\ 0 & 1 & 0 & -44 & 6 & 3 \\ 0 & 0 & 1 & 9 & 1 & 0 \end{array}\right]$$

$$P_{T \leftarrow S} = \begin{bmatrix} 76 & -9 & -5 \\ -44 & 6 & 3 \\ 9 & 1 & 0 \end{bmatrix}$$

c) $$\left[\begin{array}{ccc|c} 5 & 7 & 1 & 2 \\ 1 & -2 & 0 & -4 \\ -3 & 4 & 1 & 15 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right], \quad [q]_S = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

d) $[q]_T = P_{T \leftarrow S} [q]_S$

$$[q]_T = \begin{bmatrix} 76 & -9 & -5 \\ -44 & 6 & 3 \\ 9 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -186 \\ 109 \\ -17 \end{bmatrix}$$

e) $$\left[\begin{array}{ccc|c} 4 & 7 & 1 & 2 \\ -3 & -5 & 1 & -4 \\ 1 & 2 & 1 & 15 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -186 \\ 0 & 1 & 0 & 109 \\ 0 & 0 & 1 & -17 \end{array}\right]$$

$\hookrightarrow [q]_T$

Question 2. Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + 2x_2 - x_3 \\ -x_1 + 4x_2 - 3x_3 \end{bmatrix}.$$

(a) What must be the value of a if $\begin{bmatrix} a \\ -2a \\ 1 \end{bmatrix} \in \mathrm{Ker}(L)$?

(b) What must be the value of a if $\begin{bmatrix} a \\ -2a \\ 1 \end{bmatrix} \in \mathrm{Range}(L)$?

(c) Find a basis for $\mathrm{Ker}(L)$.

(d) Find $\mathrm{nullity}(L)$ and $\mathrm{rank}(L)$.

(e) Is L invertible? Justify your answer.

Answer 2.

c) $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathrm{Ker}L$ iff $L(u)=0$, $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 0$,

$x_1 - x_2 + x_3 = 0$
$x_1 + 2x_2 - x_3 = 0$
$-x_1 + 4x_2 - 3x_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{1} & 0 & 1/3 \\ 0 & \textcircled{1} & -2/3 \\ 0 & 0 & 0 \end{bmatrix},$$

$x_3 = t$
$x_1 = -1/3\,t$
$x_2 = 2/3\,t$

$$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/3 \\ 2/3 \\ 1 \end{bmatrix}}_{\text{basis for } \mathrm{Ker}L} t, \quad t \in \mathbb{R}$$

a) $$\left.\begin{aligned} -\tfrac{1}{3}t &= a \\ 2/3\,t &= -2a \\ t &= 1 \end{aligned}\right\} \Rightarrow a = -1/3$$

b) $\begin{bmatrix} a \\ -2a \\ 1 \end{bmatrix} \in \mathrm{Range}L \Rightarrow$

$x_1 - x_2 + x_3 = a$
$x_4 + 2x_2 - x_3 = -2a$
$-x_1 + 4x_2 - 3x_3 = 1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 2 & -1 & -2a \\ -1 & 4 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 3 & -2 & -3a \\ 0 & 0 & 0 & \underbrace{4a+1} \end{array}\right]$$

$4a+1 = 0 \Rightarrow a = -1/4$

d) $\mathrm{nullity}(L) = 1$, $\mathrm{rank}(L) = 2$

e) NO, $\mathrm{Ker}L \neq \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

Question 3. Let $q_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $q_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\alpha = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ and W be a subspace of $\mathbb{R}^3$ spanned by q_1 and q_2.

(a) Find an orthogonal basis for W.

(b) Find Proj_W^{α}.

(c) Find a basis for $W^{\perp}$.

(d) Find $\text{Proj}_{W^{\perp}}^{\alpha}$.

(e) Write $\alpha = \beta_1 + \beta_2$ where $\beta_1 \in W$ and $\beta_2 \in W^{\perp}$.

Answer 3.

a) $S = \{q_1 = (1,2,1),\ q_2 = (2,1,-1)\}$ linearly indep. set.

Using G.S.O.P

$$\alpha_1 = q_1 = (1,2,1)$$

$$\alpha_2 = q_2 - \frac{\langle q_2, \alpha_1 \rangle}{\|\alpha_1\|^2} . \alpha_1 = (2,1,-1) - \frac{\langle (2,1,-1),(1,2,1) \rangle}{\|(1,2,1)\|^2} . (1,2,1)$$

$$\alpha_2 = (\tfrac{3}{2}, 0, -3/2)$$

$T = \{\alpha_1 = (1,2,1),\ \alpha_2 = (3/2, 0, -3/2)\}$ orthogonal basis for W.

b) $$\text{Proj}_W^{\alpha} = \frac{\langle \alpha, \alpha_1 \rangle}{\|\alpha_1\|^2} . \alpha_1 + \frac{\langle \alpha, \alpha_2 \rangle}{\|\alpha_2\|^2} . \alpha_2$$

$$\text{Proj}_W^{\alpha} = \frac{\langle (2,1,5),(1,2,1) \rangle}{\|(1,2,1)\|^2} . (1,2,1) + \frac{\langle (2,1,5),(3/2,0,-3/2) \rangle}{\|(\frac{3}{2},0,-\frac{3}{2})\|^2} . (\tfrac{3}{2}, 0, -\tfrac{3}{2})$$

$$\text{Proj}_W^{\alpha} = (0,3,3) = \beta_1$$

d) $\text{Proj}_{W^{\perp}}^{\alpha} = \alpha - \text{Proj}_W^{\alpha} = (2,1,5) - (0,3,3) = (2,-2,2) = \beta_2$

e) $\alpha = \text{Proj}_W^{\alpha} + \text{Proj}_{W^{\perp}}^{\alpha} = \beta_1 + \beta_2$

$\alpha = (0,3,3) + (2,-2,2)$

c) If W is the row space of A then $W^{\perp}$ is the nullspace of A,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} ① & 0 & -1 \\ 0 & ① & 1 \end{bmatrix} \quad \begin{array}{l} x_3 = t \\ x_1 = t \\ x_2 = -t \end{array}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\text{basis for } W^{\perp}} t, \quad t \in \mathbb{R}$$

Question 4. Let

$$A = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -3 \end{bmatrix}.$$

(a) Find the eigenvalues and the corresponding eigenvectors of A.

(b) Find the determinant of A. Is A invertible?

(c) Find, if possible, an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

(d) Find the eigenvalues of A^{-1}.

(e) Find the eigenvalues and the associated eigenvectors of A^7.

Answer 4.

a) $\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & 0 & 1 \\ 0 & \lambda+2 & 0 \\ 1 & 0 & \lambda+3 \end{vmatrix} = (\lambda+2)^2 . (\lambda+4) = 0$, $\lambda_1 = -2$, $\lambda_2 = -2$, $\lambda_3 = -4$

$\lambda_2 = \lambda_1 = -2 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = t$, $x_3 = s$, $x_1 = -s$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t$, $s, t \in \mathbb{R}$

$P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ eigenvectors corr. to $\lambda_1 = -2$

$\lambda_3 = -4 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_3 = t$, $x_1 = t$, $x_2 = 0$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$, $t \in \mathbb{R}$

$P_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ eigenvector corr. to $\lambda_2 = -4$

c) Since $A^T = A$, A is symmetric, there exists an orthogonal matrix Q.

$\langle P_1, P_2 \rangle = 0$, $\langle P_1, P_3 \rangle = 0$, $\langle P_2, P_3 \rangle = 0$, they are orthogonal.

$q_1 = \frac{P_1}{\|P_1\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $q_2 = \frac{P_2}{\|P_2\|} = \frac{(0,1,0)}{1} = (0,1,0)$, $q_3 = \frac{P_3}{\|P_3\|} = \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$Q = \left[q_1 \,\vdots\, q_2 \,\vdots\, q_3 \right] = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

b) $\det(A) = \lambda_1 . \lambda_2 . \lambda_3 = (-2).(-2).(-4) = -16 \neq 0$, Yes.

d) $(-2)^{-1} = -\frac{1}{2}$ eigenvalue of A^{-1}

$(-4)^{-1} = -\frac{1}{4}$ " " A^{-1}

e) $(-2)^7$ eigenvalue of A^7, and $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ eigenvectors of A^7 corr. to $(-2)^7$

$(-4)^7$ eigenvalue of A^7 and $P_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ eigenvector of A^7 corr. to $(-4)^7$

Question 5. Let

$$A = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix}.$$

(a) Show that A is Hermitian.
(b) Find a unitary matrix U that diagonalize A.

Answer 5.

a) $\bar{A} = \begin{bmatrix} 0 & 3-4i \\ 3+4i & 0 \end{bmatrix}$, $(\bar{A})^T = A^* = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix} = A \Rightarrow A$ is Hermitian.

b) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -3-4i \\ -3+4i & \lambda \end{vmatrix} = \lambda^2 - 25 = (\lambda-5)(\lambda+5) = 0$

$\lambda_1 = 5, \ \lambda_2 = -5$

For $\lambda_1 = 5$, $\begin{bmatrix} 5 & -3-4i \\ -3+4i & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{-3-4i}{5} \\ 0 & 0 \end{bmatrix}$ $\quad x_2 = t$

$x_1 = \frac{3+4i}{5} t$

$X = \underbrace{\begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix}}_{\to P_1} t$

For $\lambda_2 = -5$, $\begin{bmatrix} -5 & -3-4i \\ -3+4i & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3+4i}{5} \\ 0 & 0 \end{bmatrix}$ $\quad x_2 = t$

$x_1 = -\left(\frac{3+4i}{5}\right) t$

$X = \underbrace{\begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix}}_{\to P_2} t$

$\|P_1\| = \sqrt{2}$, $\quad \|P_2\| = \sqrt{2}$, $\quad u_1 = \frac{P_1}{\|P_1\|}$, $\quad u_2 = \frac{P_2}{\|P_2\|}$

$U = [u_1 \vdots u_2] = \begin{bmatrix} \frac{3+4i}{5\sqrt{2}} & -\frac{(3+4i)}{5\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ unitarily diagonalize A and

$U^{-1} A U = U^* A U = D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

$(U^{-1} = U^*)$

Question 6. Let A be an $n \times n$ complex matrix and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Show that

(a) if A is Hermitian then $\bar{\lambda} = \lambda$,

(b) if A is unitary then $\bar{\lambda} = \dfrac{1}{\lambda}$.

Answer 6.

a) $AX' = \lambda X'$ iff $A^*X = \bar{\lambda}X'$

Since A is Hermitian, $A^* = A$ implies that for an eigenvalue λ

$\lambda X' = AX' = A^*X' = \bar{\lambda}X'$ with $X' = 0$

$\Rightarrow \lambda = \bar{\lambda}$

b) A is unitary, $A^* = A^{-1}$

$AX' = \lambda X'$ iff $A^*X' = \bar{\lambda}X'$ and

$$X' = \underbrace{A^*AX'}_{\text{A is unitary}} = A^*(AX') = A^*(\lambda X) = \lambda A^*X' = \lambda.\bar{\lambda}X'$$

$\Rightarrow \lambda.\bar{\lambda} = 1 \Rightarrow \bar{\lambda} = \dfrac{1}{\lambda}$

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MCS 231 Linear Algebra
Make-up
January 24, 2011, 15:00-17:00

QUESTIONS

(1) Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

(a) Find an orthogonal basis S for the subspace W of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3\}$.

(b) Find an orthonormal basis T for the subspace W of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3\}$.

(c) Find $[v]_S$ if $v = \begin{bmatrix} 0 \\ -1 \\ 4 \\ 5 \end{bmatrix}$.

(d) Find a basis for $W^\perp$.

(2) Describe the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ which is represented by the matrix

$$A = \begin{bmatrix} -2 & 3 & 1 \\ 5 & -6 & 0 \end{bmatrix}$$

with respect to the standard basis for $\mathbb{R}^3$ and the basis $\left\{ \begin{bmatrix} 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

(3) Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear operator on defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ x_1 + 2x_3 - x_4 \\ -3x_2 + 2x_4 \end{bmatrix}.$$

(a) Find a basis β for $\mathrm{Ker}(L)$.

(b) Extend β to basis for $\mathbb{R}^4$.

(c) Find a basis for $\mathrm{Range}(L)$.

(d) Is $v = \begin{bmatrix} 15 \\ -2 \\ 19 \end{bmatrix} \in \mathrm{Range}(L)$?

(e) Find $\mathrm{nullity}(L)$ and $\mathrm{rank}(L)$.

(4) Let

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

- (a) Find the eigenvalues and the corresponding eigenvectors of A.
- (b) Find the determinant of A. Is A invertible?
- (c) Find, if possible, an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.
- (d) Find the eigenvalues and the associated eigenvectors of A^{17}.

(5) Let

$$A = \begin{bmatrix} -2 & 1+2i & 0 \\ 1-2i & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Show that A is Hermitian.
- (b) Find a unitary matrix U that diagonalize A.

(6) (a) Is the following matrix orthogonal? Justify your answer.

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

(b) For any complex square matrix A, show that $A + A^*$ is Hermitian.